Geometric equations of state in Friedmann-Lemaître universes admitting Matter and Ricci Collineations

Pantelis S. Apostolopoulos and Michael Tsamparlis

Department of Physics, Section of Astrophysics-Astronomy-Mechanics,

University of Athens, Zografos 15783, Athens, Greece

February 7, 2008

Abstract

As a rule in General Relativity the spacetime metric fixes the Einstein tensor and through the Field Equations (FE) the energy-momentum tensor. However one cannot write the FE explicitly until a class of observers has been considered. Every class of observers defines a decomposition of the energy-momentum tensor in terms of the dynamical variables energy density (μ), the isotropic pressure (p), the heat flux q^a and the traceless anisotropic pressure tensor π_{ab}. The solution of the FE requires additional assumptions among the dynamical variables known with the generic name equations of state. These imply that the properties of the matter for a given class of observers depends not only on the energy-momentum tensor but on extra a priori assumptions which are relevant to that particular class of observers. This makes difficult the comparison of the Physics observed by different classes of observers for the *same* spacetime metric. One way to overcome this unsatisfactory situation is to define the extra condition required among the dynamical variables by a geometric condition, which will be based on the metric and not to the observers. Among the possible and multiple conditions one could use the consideration of collineations. We examine this possibility for the Friedmann-Lemaître-Robertson-Walker models admitting matter and Ricci collineations and determine the equations of state for the comoving observers. We find linear and non-linear equations of state, which lead to solutions satisfying the energy conditions, therefore describing physically viable cosmological models.

KEY WORDS: Matter Collineations; Ricci Collineations; Robertson-Walker spacetimes; Equations of state;

1 Introduction

In General Relativity one usually restricts the spacetime metric by means of some symmetry conditions. The metric fixes the Einstein tensor, and through Field Equations (FE), the energy-momentum tensor T_{ab}. To make Physics one has to define physical quantities. This is done by the consideration of a unit timelike vector field, u^a say, which physically is identified with the

field of observers and geometrically it is used to 1+3 decompose T_{ab} (and any other field) in the well known way [1]:

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2q_{(a}u_{b)} + \pi_{ab}. \tag{1}$$

Geometrically the quantities μ, p, q_a, π_{ab} are tensors, which depend on *both* T_{ab} and u^a. Physically these quantities are the dynamical variables (energy density, isotropic pressure, heat flux and traceless anisotropic pressure) of the considered spacetime as observed by the *specific* observers u^a.

The vector field u^a preferably is defined from the given spacetime metric (or energy-momentum tensor) by means of some characteristic geometric property. For example the most standard class of observers are the comoving observers *defined* by the timelike eigenvector of the Ricci (or equivalently the energy-momentum) tensor. For these observers $q^a = 0$, that is, the heat conduction vector vanishes[1]. Of course it is possible that a given spacetime metric gives rise to more than one geometrically defined timelike vector fields, in which case one has potentially many inherent classes of observers.

One well known example is furnished by the spatially homogeneous tilted perfect fluid comologies in which it is possible to consider two future directed timelike unit vector fields [2]. The normal vector n^a to the hypersurfaces of homogeneity and the eigenvector u^a of the Ricci tensor. These vectors are not necessarily parallel and define the so called hyperbolic angle of tilt β by the relation $\cosh\beta = -u^a n_a$. When $\beta \neq 0$ each of these vector fields defines a class of observers. The observers u^a are the comoving ones and the vector n^a is not an eigenvector of the energy-momentum tensor and the matter is that of an "imperfect fluid with particular equations of state" (equation (1.33a) of [2]).

In case the metric (or any other relevant object) does not have more than one inherent timelike vector fields, these can be introduced, if it is considered necessary, by means of additional requirements which can be either geometric or "physical".

For example the Friedmann-Lemaître-Robertson-Walker (FLRW) model due to its high symmetry allows only one characteristic (unit) timelike vector, which is the timelike eigenvector of the Ricci tensor. Coley and Tupper introduced tilted observes in flat FLRW models by the (physical) requirement that they lead to magnetohydrodynamic viscous fluid solutions with heat conduction [3]. One can find many similar examples in the literature.

When we write the FE in terms of the physical variables of a certain class of observers we find a system of equations, which is not closed. This necessitates the introduction of *additional* equation(s) among the physical variables known as "equation(s) of state" or more general as "constitutive equations". For perfect fluids the standard equations of state are relations of the form $p = p(\mu)$. It is important to note that these additional "contstraint" equations are defined (a) in an a priori manner and (b) they hold *only* for the specific dynamical variables, that is, the class of observers, they refer to. Therefore the Physics of a given spacetime of two different classes of observers cannot be compared!

One way out of this unsatisfactory situation is to consider that the required additional equation(s) of state will be defined by means of a geometric requirement/condition on the metric, therefore it will be common to all classes of observers for that metric. In practice this can be realized as follows. The condition on the metric is expressed by a set of equations

[1] The anisotropic pressure tensor π_{ab} does not necessarily vanish therefore the matter observed by the comoving observers is not necessarily a perfect fluid.

involving the metric components and possibly some other relevant parameters. For any class of observers we express the geometric condition in terms of the physical variables of these observers and the resulting equations we consider to be the equation(s) of state (of *that* metric for *these* observers).

In the case the geometric conditions are more than the required number of equations one can define, for each appropriate subset of them, a corresponding equation of state.

For another class of observers the *same* geometric condition when written in terms of their physical variables will provide, in general, different equation(s) of state. The important aspect is that the differences between the equations of state will be due to the differences in the observers only. Eventually what we have gained is:

a. A way to produce equations of state consistent with the geometry of spacetime

b. A common ground on which we can compare the equations of state of different classes of observers.

One might argue that we "let Geometry do the Physics" and that physical intuition is lost. But this is a common ground in General Relativity, where we can have a "direct" physical picture only for the simplest cases. The proposition we make gives more weight to the geometric consistency of the (simplifying) assumptions involved and less to the physical intuition, and in any case, it can be seen as a "phenomenological" approach, which, however, is geometrically sound.

We have still to discuss the nature of the geometric condition on the metric. Obviously there are many possible alternatives and, furthermore, there is no a priori guarantee that whichever is considered will lead to equation(s) of state, which will be of physical interest.

In this paper we propose that the condition is that the metric admits a higher collineation (a term to be defined in section 2). This proposal is logical for many reasons. Indeed one could argue that we believe the symmetry at the level of the metric (the KVs) therefore there is no reason why we should not believe it in its higher levels (i.e collineations).

The interplay between collineations and equations of state is not new. For example McIntosh [4] has shown that if in any spacetime, in which we consider a class of observers such that the matter is a perfect fluid, we require that there exists a non-trivial HVF which is normal to the 4-velocity of the observers, then the equation of state must be $p = \mu$.[2]

In the next sections we apply the above analysis to the following set:

a. The FLRW spacetimes,

b. The comoving observers for this metric i.e $u^a = \delta^a_0$,

c. The geometric condition is the requirement that the metric admits a Ricci or a matter collineation.

As it will be shown these assumptions lead to solutions of the FE, which satisfy the energy conditions, therefore they are physically meaningful and lead to interesting results.

The structure of the paper is as follows. In section 2 we discuss briefly higher collineations. In section 3 we present in a concise manner the Ricci and matter collineations and give the resulting algebraic constraints on the spatial components of the Ricci and matter tensor. In section 4 we determine and study the equations of state of the resulting FLRW models for the standard comoving observers. Finally in section 5 we draw our conclusions.

[2] Wainwright [5] generalized this result and showed that if in a spacetime the observers are chosen so that there is an equation of state $p = p(\mu)$ then if we demand further that the spacetime admits a proper HVF, the equation of state is reduced to the form $p = \alpha\mu$ where α is a constant.

2 General comments on collineations

A geometric symmetry or collineation is defined by a relation of the form:

$$\mathcal{L}_\xi \mathbf{\Phi} = \mathbf{\Lambda} \tag{2}$$

where ξ^a is the symmetry or collineation vector, $\mathbf{\Phi}$ is any of the quantities $g_{ab}, \Gamma^a_{bc}, R_{ab}, R^a_{bcd}$ and geometric objects constructed from them and $\mathbf{\Lambda}$ is a tensor with the same index symmetries as $\mathbf{\Phi}$. By demanding specific forms for the quantities $\mathbf{\Phi}$ and $\mathbf{\Lambda}$ one finds all the well known collineations. For example $\Phi_{ab} = g_{ab}$ and $\Lambda_{ab} = 2\psi g_{ab}$ defines the conformal Killing vector (CKV) and specializes to a special conformal Killing vector (SCKV) when $\psi_{;ab} = 0$, to a homothetic vector field (HVF) when $\psi =$constant and to a Killing vector (KV) when $\psi = 0$. When $\Phi_{ab} = R_{ab}$ and $\Lambda_{ab} = 2\psi R_{ab}$ the symmetry vector ξ^a is called a Ricci conformal collineation (RCC) and specializes to a Ricci collineation (RC) when $\Lambda_{ab} = 0$. When $\Phi_{ab} = T_{ab}$ and $\Lambda_{ab} = 2\psi T_{ab}$, where T_{ab} is the energy momentum tensor, the vector ξ^a is called a matter conformal collineation (MCC) and specializes to a matter collineation (MC) when $\Lambda_{ab} = 0$. The function ψ in the case of CKVs is called the conformal factor and in the case of conformal collineations the *conformal function.*

Collineations of a different type are not necessarily independent, for example a KV or a HVF is a RC or a MC but not the opposite. A RC or a MC which is not a KV or a HVF (and in certain cases a SCKV) is called proper. There are many types of collineations and most of them have been classified in a diagram which exhibits clearly their relative properness [6, 7].

Although much work has been done on the computation of the higher collineations for various types of metrics, comparatively little has been done towards their applications in General Relativity. These applications are usually in the direction of conservation laws. For example it has been shown that a CKV generates a constant of motion along a null geodesic. This result has been used to solve completely Liouville's equation for photons in FLRW spacetimes [8, 9]. Furthermore timelike RCs have been related to the conservation of particle number in FLRW spacetimes [10].

3 The RCs and MCs of the FLRW models

The geometry of the FLRW models is described by the Robertson-Walker metric which in the standard coordinates is[3]:

$$ds_g^2 = g_{ab}dx^a dx^b = S^2(\tau)\left[-d\tau^2 + U^2(k, x^\alpha)d\sigma_3^2\right] \tag{3}$$

where $U(k, x^\alpha) = \left(1 + \frac{k}{4}x^\alpha x_\alpha\right)^{-1}$, $k = \frac{R}{12}$, $d\sigma_3^2 = dx^2 + dy^2 + dz^2$ and R is the scalar curvature of the spatial hypersurfaces. τ is the conformal time along the world line of the comoving observers $u^a = S^{-1}\delta^a_\tau$ and it is related to the standard variable t (cosmic time in FLRW spacetime) with the relation:

$$dt = S(\tau)d\tau. \tag{4}$$

[3]Throughout the paper the index convention is such that latin indices $a, b, c, ...$ take the values $0, 1, 2, 3$ whereas Greek indices $\alpha, \beta, \gamma ... = 1, 2, 3$.

Provided the Ricci tensor R_{ab} and the matter tensor T_{ab} of (3) are non-degenerate they define the Ricci and the matter "metric" ds_R^2, ds_T^2 respectively by the following expressions:

$$ds_R^2 = R_{ab}dx^a dx^b = R_0(\tau)d\tau^2 + R_1(\tau)U^2(k, x^\alpha)d\sigma_3^2 \tag{5}$$

where:

$$R_0(\tau) = \frac{3\left[(S_{,\tau})^2 - SS_{,\tau\tau}\right]}{S^2} \tag{6}$$

$$R_1(\tau) = \frac{SS_{,\tau\tau} + (S_{,\tau})^2 + 2kS^2}{S^2}. \tag{7}$$

and

$$ds_T^2 = T_{ab}dx^a dx^b = T_0(\tau)d\tau^2 + T_1(\tau)U^2(x^\alpha)d\sigma_3^2 \tag{8}$$

where:

$$T_0 = 3\frac{(S_{,\tau})^2 + kS^2}{S^2}, T_1 = \frac{-2SS_{,\tau\tau} + (S_{,\tau})^2 - kS^2}{S^2} \tag{9}$$

To compute the RCs and the MCs of (3) we observe that the three line elements ds_g^2, ds_R^2, ds_T^2 (not in general of the same signature!) have the same functional form, that is, all of them can be constructed form the conformally flat "generic" line element:

$$ds^2 = K_{ab}dx^a dx^b = A^2(\tau_A)\left[\epsilon d\tau_A^2 + U^2(k, x^\alpha)d\sigma_3^2\right] \tag{10}$$

where the function $U(k, x^\alpha)$ has been defined above, $\epsilon = \pm 1$ for appropriate choices of the function $A(\tau_A)$ and the zero coordinate $\tau_A = \int \left|\frac{K_0}{K_1}\right| d\tau$. Therefore the RCs and the MCs (for the case R_{ab}, T_{ab} are non-degenerate[4]) are found from the KVs of the metric ds^2 provided one replaces the appropriate expressions for the metric components $A(\tau_A)$. This has been done in [11, 12] and independently in [13]. For completeness we summarize the results of these works in Tables 1-4. The quantities $c_\pm, \phi_k$ are defined in Table 5.

TABLE 1. The proper RCs of the FLRW spacetimes for $k = \pm 1$ and the expression of R_1 for which the corresponding collineations are admitted. A is an integration constant. Furthermore we define $\hat{\tau} = \int |R_0|^{1/2} d\tau$.

#	RCs $\mathbf{X}$ $(k = \pm 1)$	$\mathbf{R}_1(\tau)$
1	$\mathbf{Y} = A\partial_{\hat{\tau}}$	A
4	$\mathbf{H}_1^k = \epsilon k \phi_k(\mathbf{H})A\partial_{\hat{\tau}} + \mathbf{H}ta_{-\epsilon k}(\frac{\hat{\tau}}{A})$ $\mathbf{Q}_\mu^k = \epsilon k \phi_k(\mathbf{C}_\mu)A\partial_{\hat{\tau}} + \mathbf{C}_\mu ta_{-\epsilon k}(\frac{\hat{\tau}}{A})$	$\epsilon A^2 c_{-\epsilon k}^2(\frac{\hat{\tau}(\tau)}{A})$
4	$\mathbf{H}_2^\epsilon = \phi_\epsilon(\mathbf{H})A\partial_{\hat{\tau}} - \mathbf{H}\coth(\frac{\hat{\tau}}{A})$ $\mathbf{Q}_{\mu+3}^k = \phi_k(\mathbf{C}_\mu)A\partial_{\hat{\tau}} - \mathbf{C}_\mu\coth(\frac{\hat{\tau}}{A})$	$\epsilon A^2 \sinh^2(\frac{\hat{\tau}(\tau)}{A})$

[4]In case where R_{ab}, T_{ab} are degenerate the Lie algebra of RCs and MCs is infinite dimensional and must be found by the solution of the FE. However, because they are infinite dimensional, they are not as useful as the ones of the non-degenerate case.

TABLE 2. The proper RCs of the FLRW spacetimes for $k = 0$. A is an integration constant.

#	RCs $\mathbf{X}$ ($k = 0$)	$\mathbf{R}_1(\tau)$
4	$\mathbf{P}_{\hat{\tau}} = \lvert A\rvert^{1/2}\,\partial_{\hat{\tau}}$ $\mathbf{M}_{\alpha\hat{\tau}} = x^{\alpha}\,\lvert A\rvert^{1/2}\,\partial_{\hat{\tau}} - \epsilon\hat{\tau}(\tau)\,\lvert A\rvert^{-1/2}\,\partial_{\alpha}$	A
4	$\mathbf{H} = A\partial_{\hat{\tau}} + x^{\alpha}\partial_{\alpha}$ $\mathbf{K}_{\alpha} = 2x_{\alpha}\mathbf{H} - \left(\epsilon e^{2\hat{\tau}/A} + x^{\beta}x_{\beta}\right)\partial_{\alpha}$	$\epsilon A^2 e^{-2\hat{\tau}(\tau)/A}$

TABLE 3. The proper MCs of the FLRW spacetimes for $k = \pm 1$ and the expression of T_1 for which MCs are admitted. A is an integration constant. Furthermore we define $\tilde{\tau} = \int \lvert T_0\rvert^{1/2} d\tau$.

#	MCs $\mathbf{X}$ ($k = \pm 1$)	$\mathbf{T}_1(\tau)$
1	$\mathbf{Y} = A\partial_{\tilde{\tau}}$	A
4	$\mathbf{H}_1^k = \epsilon k\phi_k(\mathbf{H})A\partial_{\tilde{\tau}} + \mathbf{H}ta_{-\epsilon k}(\frac{\tilde{\tau}}{A})$ $\mathbf{Q}_{\mu}^k = \epsilon k\phi_k(\mathbf{C}_{\mu})A\partial_{\tilde{\tau}} + \mathbf{C}_{\mu}ta_{-\epsilon k}(\frac{\tilde{\tau}}{A})$	$\epsilon A^2 c^2_{-\epsilon k}(\frac{\tilde{\tau}(\tau)}{A})$
4	$\mathbf{H}_2^{\epsilon} = \phi_{\epsilon}(\mathbf{H})A\partial_{\tilde{\tau}} - \mathbf{H}\coth(\frac{\tilde{\tau}}{A})$ $\mathbf{Q}_{\mu+3}^k = \phi_k(\mathbf{C}_{\mu})A\partial_{\tilde{\tau}} - \mathbf{C}_{\mu}\coth(\frac{\tilde{\tau}}{A})$	$\epsilon A^2\sinh^2(\frac{\tilde{\tau}(\tau)}{A})$

TABLE 4. The proper MCs of the FLRW spacetimes for $k = 0$. A is an integration constant.

#	RCs $\mathbf{X}$ ($k = 0$)	$\mathbf{T}_1(\tau)$
4	$\mathbf{P}_{\tilde{\tau}} = \lvert A\rvert^{1/2}\,\partial_{\tilde{\tau}}$ $\mathbf{M}_{\alpha\tilde{\tau}} = x^{\alpha}\,\lvert A\rvert^{1/2}\,\partial_{\tilde{\tau}} - \epsilon\tilde{\tau}(\tau)\,\lvert A\rvert^{-1/2}\,\partial_{\alpha}$	A
4	$\mathbf{H} = A\partial_{\tilde{\tau}} + x^{\alpha}\partial_{\alpha}$ $\mathbf{K}_{\alpha} = 2x_{\alpha}\mathbf{H} - \left(\epsilon e^{2\tilde{\tau}/A} + x^{\beta}x_{\beta}\right)\partial_{\alpha}$	$\epsilon A^2 e^{-2\tilde{\tau}(\tau)/A}$

TABLE 5. The quantities $\phi, c_{\pm}, s_{\pm}$ which appear in Tables 1-4.

ϕ	$\mathbf{c}_{\pm}, \mathbf{s}_{\pm}$
$\phi_k(\mathbf{H}) = 1 - \frac{kU\cdot(x^{\sigma}x_{\sigma})}{2}$	$(c_+, c_-) \equiv [\cosh\tilde{\tau}(\tau), \cos\tilde{\tau}(\tau)]$
$\phi_k(\mathbf{C}_{\mu}) = -kUx_{\nu}$	$(s_+, s_-) \equiv [\sinh\tilde{\tau}(\tau), \sin\tilde{\tau}(\tau)]$

4 Geometric equations of state

Consider the standard FLRW model with vanishing cosmological constant and comoving observers $u^a = S^{-1}(\tau)\delta^a_{\tau}$, where $\tau = \int \frac{dt}{S(t)}$, t being the standard cosmic time. For these observers the energy-momentum tensor has a perfect fluid form i.e. $T_{ab} = \mu u_a u_b + p h_{ab}$ where μ, p are the energy density and the isotropic pressure measured by the observers u^a. This decomposition of T_{ab} in the coordinates (τ, x^{α}) leads to the relations:

$$T_{00} = \mu S^2(\tau), T_{11} = T_{22} = T_{33} = pS^2(\tau)U^2(k, x^{\alpha}). \tag{11}$$

Expression (11) is a result (a) of the symmetry assumptions of the metric and (b) the choice of observers. Using FE we compute the spatial components of the Einstein tensor $T_{00}(S, S_{,\tau}, S_{,\tau\tau}, U)$,

$T_{11}(S, S_{,\tau}, S_{,\tau\tau}, U)$ in terms of the scale factor $S(\tau)$ and its derivatives. From (8), (9) and (11) it follows:

$$\mu = 3\frac{(S_{,\tau})^2 + kS^2}{S^4}, p = \frac{-2SS_{,\tau\tau} + (S_{,\tau})^2 - kS^2}{S^4} \tag{12}$$

There remains one variable (the $S(\tau)$) free. Therefore we have to supply one more equation in order to solve the model. This extra equation is a barotropic equation of state, that is, a relation of the form $p = p(\mu)$.

The obvious choice is a linear equation of state $p = (\gamma - 1)\mu$. There are several solutions for this simple choice which are of cosmological interest. For example $\gamma = 1$ ($p = 0$) implies degeneracy of the energy momentum-tensor (dust) and the value $\gamma = \frac{4}{3}$ ($p = \frac{1}{3}\mu$) implies radiation dominated matter. Both states of matter are extreme and they have been relevant at certain stages of the evolution of the Universe. For other values of γ one obtains intermediate states which cannot be excluded as unphysical (see e.g. [14] for a thorough review). Therefore it would be interesting to use non-linear equations of state which will deal with more complex-and physical-forms of matter. But what will be an "objective" criterion to write such equations?

We propose that this equation will be one of the constraint equations defined by the requirement of existence of a proper RC or a MC. Of course for every choice of observers every such equation will have a different form, which will have to be checked that it leads to physically reasonable results. From the geometric point of view this proposal is reasonable. Indeed the KVs are used to fix the general form of the metric and, because a KV is a RC and a MC, they also fix the R_{ab}, T_{ab}. Therefore the proper RCs and MCs are symmetries which contain the effects of covariant differentiation (R_{ab}) and FE (T_{ab}) therefore it is reasonable to expect that they will have immediate and stronger physical implications. One extra advantage of this type of equations of state is that, unlikely the standard ones, they are observer independent in the sense that they take a specific form only after a class of observers is selected. We call these conditions *geometric equations of state*.

We recall that the two important kinematic quantities in the FLRW universe are the Hubble scalar H and the deceleration parameter q, which are defined as follows:

$$H = \frac{1}{3}\theta = \frac{S_{,\tau}}{S^2} \quad , \quad q = 1 - \frac{SS_{,\tau\tau}}{(S_{,\tau})^2}. \tag{13}$$

In the following we demonstrate the above considerations for the case of RCs and MCs in FLRW models and determine the equations of state for the comoving observers. As we have mentioned in the introduction the set of equations we shall need follows from the symmetry conditions, the FE and the conservation equation. For the case of the FLRW cosmological models these are:

$$f(\mu, p, \mu_{,\tau}, p_{,\tau}) = 0 \qquad \text{(Symmetry condition)} \tag{14}$$

$$\mu_{,\tau} = -3H(\mu + p)S \qquad \text{(Bianchi identity)} \tag{15}$$

$$H = \frac{\sqrt{3}}{3}(\mu - \frac{3k}{S^2})^{1/2} \qquad \text{(Friedmann equation).} \tag{16}$$

4.1 Geometric equations of state for MCs

For convenience we define the new the "time" coordinate $\tilde{\tau}$ in terms of the energy density of the fluid as follows:

$$\tilde{\tau}(\tau) = \int |T_0(\tau)|^{1/2} d\tau = \int \left(3\frac{(S_{,\tau})^2 + kS^2}{S^2} \right)^{1/2} d\tau = \int \sqrt{\mu} S d\tau. \tag{17}$$

Case A. $k = 0$

From Table 4 we see that there are two cases to consider i.e. $T_1 = A$ =constant and $T_1 = \epsilon A^2 e^{-2\tilde{\tau}(\tau)/A}$. This means that one can define two families of geometric equations of state.

Case AI: $T_1(\tau) = A \equiv \varepsilon_1 a^2$ $(\varepsilon_1 = \pm 1, a \in R)$

The constraint $T_1(\tau) = \varepsilon_1 a^2$ (where a is a constant) leads to the condition:

$$pS^2(\tau) = \varepsilon_1 a^2 \tag{18}$$

which by means of the second of (12) gives the equation:

$$-2SS_{,\tau\tau} + (S_{,\tau})^2 = \varepsilon_1 a^2 S^2. \tag{19}$$

The solution of (19) provides the unknown scale factor $S(\tau)$ and describes the FLRW model completely. To solve equation (19) we write it in the form:

$$2\left(\frac{S_{,\tau}}{S}\right)_{,\tau} + \left(\frac{S_{,\tau}}{S}\right)^2 = -\varepsilon_1 a^2 \tag{20}$$

which can be integrated easily. In Table 6 we give all four solutions of (19) together with the physical variables of the model that is, energy density (μ), isotropic pressure (p), Hubble scalar (H) and deceleration parameter (q).

TABLE 6. The FRW models with $k = 0$ which admit the MCs $P_{\hat{\tau}}, M_{\mu\hat{\tau}}$ and B, C are arbitrary integration constants.

Case	$S(\tau)$	$\mu(\tau)$	$p(\tau)$	$H(\tau)$	$q(\tau)$	Restrictions
1	$Ce^{a\tau}$	$-\frac{3A}{C^2e^{2a\tau}}$	$\frac{A}{C^2e^{2a\tau}}$	$\frac{3a}{Ce^{a\tau}}$	0	$\varepsilon_1 = -1, A < 0$
2	$B\cos^2\frac{a\tau}{2}$	$\frac{12A(1-\cos a\tau)}{B^2(1+\cos a\tau)^3}$	$\frac{4A}{B^2(1+\cos a\tau)^2}$	$\frac{2a\sin a\tau}{B(1+\cos a\tau)^2}$	$\frac{1+\cos a\tau}{\sin^2 a\tau}$	$\varepsilon_1 = 1, A > 0$
3	$B\sinh^2\frac{a\tau}{2}$	$-\frac{3A}{B^2}\frac{\coth^2\frac{a\tau}{2}}{\sinh^4\frac{a\tau}{2}}$	$\frac{A}{B^2}\sinh^{-4}\frac{a\tau}{2}$	$\frac{a}{B}\frac{\coth\frac{a\tau}{2}}{\sinh^2\frac{a\tau}{2}}$	$\frac{1}{2\cosh^2 a\tau}$	$\varepsilon_1 = -1, A < 0$ $\left(\frac{S_{,\tau}}{S}\right)^2 > a^2$
4	$B\cosh^2\frac{a\tau}{2}$	$-\frac{3A}{B^2}\frac{\tanh^2\frac{a\tau}{2}}{\cosh^4\frac{a\tau}{2}}$	$\frac{A}{B^2}\cosh^{-4}\frac{a\tau}{2}$	$\frac{a}{B}\frac{\tanh\frac{a\tau}{2}}{\cosh^2\frac{a\tau}{2}}$	$-\frac{1}{2\sinh^2 a\tau}$	$\left(\frac{S_{,\tau}}{S}\right)^2 < a^2$

It is straightforward to check (e.g. by using an algebraic computing program) that indeed all four solutions of FLRW spacetimes ($k = 0$) of Table 6 admit the MCs given in Table 4. A detailed study shows that all MCs are proper, except $P_{\hat{\tau}}$ for case 1, which degenerates to a HVF. Furthermore it can be shown that $\mu > 0, \mu \pm p > 0$ and $\mu + 3p > 0$ i.e. all the energy conditions [15] are satisfied.

Concerning the determination of the equation of state[5] we use the energy conservation equation (15). The symmetry condition (18) can be written:

$$2HSp = -p_{,\tau}. \tag{21}$$

Eliminating $S(\tau)$ from (15) and (21) we find:

$$\frac{dp}{d\mu} = \frac{p_{,\tau}}{\mu_{,\tau}} = \frac{2}{3}\frac{p}{p+\mu}. \tag{22}$$

This equation has two solutions:

$$p = -\frac{1}{3}\mu \tag{23}$$

and:

$$\mu - \frac{3B}{a} \mid p \mid^{3/2} + 3p = 0. \tag{24}$$

For $B = 0$ we obtain the first solution, which is a linear equation of state with $\gamma = \frac{2}{3}$. It corresponds to the solution of case 1 of Table 6 whose metric is:

$$ds^2 = -dt^2 + t^{\frac{4}{3\gamma}}(dx^2 + dy^2 + dz^2). \tag{25}$$

This spacetime admits a HVF given by the vector $P_{\tilde{\tau}}$. The rest three vector fields are proper MCs. We note that the spacetime (25) also admits three proper RCs.

The other solution of (22) ($B \neq 0$) leads to a non-linear equation of state and corresponds to the cases 2,3,4 of Table 6.

Case AII: $T_1(\tau) = \epsilon A^2 e^{-2\tilde{\tau}(\tau)/A}$

Using the symmetry condition $T_1(\tau) = \epsilon A^2 e^{-2\tilde{\tau}(\tau)/A}$ and equations (11), (17) we find:

$$p = p_0 S^{-3B} \tag{26}$$

where $B = \frac{2}{3}\left(1 + \frac{\sqrt{3}}{A}\right)$ and $p_0 = \epsilon A^2$. Replacing this in the second equation of (12) ($k = 0$) we get:

$$2SS_{,\tau\tau} - (S_{,\tau})^2 = -p_0 S^{4-3B}. \tag{27}$$

Using (15) and (26) we can find the equation of state. Differentiating (26) we obtain:

$$p_{,\tau} = -3BpSH \tag{28}$$

which, when combined with (15), gives the following equation among the dynamic variables μ, p:

$$\frac{dp}{d\mu} = \frac{Bp}{\mu + p}. \tag{29}$$

We consider two subcases.

$B = 1 \Leftrightarrow A = 2\sqrt{3}$

The solution of (29) is:

$$\mu = p \mid \ln Cp \mid, C = const, Cp > 0 \tag{30}$$

[5] Of course we could use the expressions for $S(\tau)$ and find the same results.

which is always a non-linear equation.

<u>$B \neq 1 \Leftrightarrow A \neq 2\sqrt{3}$</u>

In this case the solution of (29) is:

$$p - Dp^{1/B} = (B-1)\mu, D = const., \; B \neq 0, 1. \tag{31}$$

Note that $D \neq 0$ is an integration constant and we have always a non-linear barotropic equation of state.

From the above we conclude that:

The only perfect fluid and flat FLRW universe with a linear equation of state, which admits proper MCs is the FLRW model (25) for $\gamma = \frac{2}{3}$.

<u>Case B: $k = \pm 1$</u>

From Table 3 we observe that the possible forms of T_1 are $T_1 = A$ and either $T_1 = \epsilon A^2 c^2_{-\epsilon k}(\frac{\tilde{\tau}(\tau)}{A})$ or $T_1 = \epsilon A^2 \sinh^2(\frac{\tilde{\tau}(\tau)}{A})$. For the later two cases equations $T_{11} = T_1 U^2(k, x^\mu)$ and (12) imply:

$$\frac{-2SS,_{\tau\tau} + (S,_{\tau})^2 - kS^2}{S^2} = \epsilon A^2 c^2_{-\epsilon k}(\frac{\tilde{\tau}(\tau)}{A}) = pS^2 \tag{32}$$

or:

$$\frac{-2SS,_{\tau\tau} + (S,_{\tau})^2 - kS^2}{S^2} = \epsilon A^2 \sinh^2(\frac{\tilde{\tau}(\tau)}{A}) = pS^2 \tag{33}$$

whose solution is difficult, due to high non-linear character of the above differential equations.

For the case $T_1 = A$ we obtain the equation:

$$T_1 = pS^2 = \frac{-2SS,_{\tau\tau} + (S,_{\tau})^2 - kS^2}{S^2} = A. \tag{34}$$

We observe that whenever $k \neq -A$ the resulting differential equation is exactly the same as in the case $k = 0$. Therefore the only interesting case is when $k = -A$ which leads to the equation:

$$\frac{-2SS,_{\tau\tau} + (S,_{\tau})^2}{S^2} = 0 \tag{35}$$

which can be written in the form:

$$\frac{-2SS,_{\tau\tau} + (S,_{\tau})^2}{S^2} = \frac{-2SS,_{\tau\tau} + 2(S,_{\tau})^2 - (S,_{\tau})^2}{S^2} = -2\left(\frac{S,_{\tau}}{S}\right)_{,\tau} - \left(\frac{S,_{\tau}}{S}\right)^2 = 0. \tag{36}$$

This equation can be solved straightforward leading to:

$$S(\tau) = S_0 \tau^2 \tag{37}$$

where S_0 is a constant of integration.

The dynamical quantities associated with the scale factor (37) are:

$$\mu = \frac{3\left(k\tau^2 + 4\right)}{S_0^2 t^6} \tag{38}$$

$$p = -\frac{k}{S_0^2 t^4}. \tag{39}$$

Eliminating t between μ, p we can find the equation of state which describe this FLRW model. Note that all the energy conditions are satisfied.

4.2 The geometric equations of state for RCs

The case of RCs is similar to that of MCs. Again we distinguish cases according to $k = 0$ and $k = \pm 1$.

Case A $k = 0$ $R_1(\tau) = A \equiv \varepsilon_1 a^2$ $(\varepsilon_1 = \pm 1, a \in R)$

From Table 2 it follows that we have to consider the following two cases.

Case AI $R_1(\tau) = A \equiv \varepsilon_1 a^2$ $(\varepsilon_1 = \pm 1, a \in R)$

The constraint $R_1(\tau) = \varepsilon_1 a^2$ leads to the condition:

$$\frac{\mu - p}{2} S^2(\tau) = \varepsilon_1 a^2. \tag{40}$$

Replacing in the expression of the Ricci tensor we find:

$$SS_{,\tau\tau} + (S_{,\tau})^2 = \varepsilon_1 a^2 S^2. \tag{41}$$

The solution of (41) provides the unknown scale factor $S(\tau)$ and describes the FLRW model completely. In Table 7 we present all three solutions of (41).

TABLE 7. The FLRW models with $k = 0$ which admit the RCs $P_{\hat{\tau}}, M_{\mu\hat{\tau}}$ and B is an arbitrary integration constant.

Case	$S(\tau)$	$\mu(\tau)$	**Restrictions**
1	$B\cos^{1/2} a\sqrt{2}\tau$	$\frac{3a^2\cos\sqrt{2}at\left(1-\cos 2\sqrt{2}a\tau\right)}{B^2\left(1+\cos 2\sqrt{2}a\tau\right)^2}$	$\varepsilon_1 = -1$
2	$B\cosh^{1/2} a\sqrt{2}\tau$	$\frac{3a^2\cosh\sqrt{2}at\left(\cosh 2\sqrt{2}a\tau-1\right)}{B^2\left(1+\cosh 2\sqrt{2}a\tau\right)^2}$	$\varepsilon_1 = 1$ $\left(\frac{S_{,\tau}}{S}\right)^2 < a^2$
3	$B\sinh^{1/2} a\sqrt{2}\tau$	$\frac{3a^2\cosh\sqrt{2}at\sinh 2\sqrt{2}a\tau}{B^2\left(1-\cosh 2\sqrt{2}a\tau\right)^2}$	$\varepsilon_1 = 1$ $\left(\frac{S_{,\tau}}{S}\right)^2 > a^2$

Similarly we determine the equation of state using equations (15) and (40). The result is:

$$(p - \mu)^3 = C(3p + \mu) \tag{42}$$

where C is a constant of integration.

Case AII $R_1(\tau) = \epsilon A^2 e^{-2\tilde{\tau}(\tau)/A}$ $(\epsilon = \pm 1, A \in \mathbf{R})$

In this case the resulting differential equation for the determination of the scale factor $S(\tau)$ is difficult and we have not been able to solve it. However we can determine the equation of state. Indeed differentiating the symmetry constraint $\left(\frac{\mu-p}{2}\right) S^2 \equiv R_1(\tau) = \epsilon A^2 e^{-2\tilde{\tau}(\tau)/A}$ we obtain:

$$\frac{(\mu - p)_{,\tau}}{\mu - p} + 2\frac{S_{,\tau}}{S} = -\frac{2}{A}\frac{d\tilde{\tau}(\tau)}{d\tau}. \tag{43}$$

The field equations imply that $R_{ab} = (\mu + p)u_a u_b + \frac{\mu-p}{2} g_{ab}$ therefore the R_0-component is $R_0 = (\mu + p)S^2$. Recalling that the new time variable $\tilde{\tau}(\tau) = \int |R_0|^{1/2} d\tau$ we rewrite (43) as:

$$\frac{(\mu - p)_{,\tau}}{\mu - p} + 2\frac{S_{,\tau}}{S} = -\frac{2}{A}(\mu + p)^{1/2} S. \tag{44}$$

Using (15) and (16) in (44) we get:

$$\frac{dp}{d\mu} = \frac{\mu + 5p}{3(\mu + p)} + \frac{2\sqrt{3}}{3}\frac{p - \mu}{[\mu(\mu + p)]^{1/2}}. \tag{45}$$

The solution of equation (45) in implicit form is:

$$\begin{aligned} &(p-\mu)^3 \left(\frac{2\sqrt{3}\sqrt{\mu(\mu+p)}+\mu+3p}{11\mu^2+6\mu p-9p^2}\right)^4 \left(\frac{\sqrt{3}\sqrt{\mu^{-1}(\mu+p)}+\sqrt{3}-1}{\sqrt{3}\sqrt{\mu^{-1}(\mu+p)}-\sqrt{3}-1}\right)^{7\sqrt{3}/3} \times \\ &\times \left[\frac{\mu\left(\sqrt{\mu^{-1}(\mu+p)}-\sqrt{2}\right)^2}{p-\mu}\right]^{3\sqrt{6}/2} = \mu_0 \end{aligned} \tag{46}$$

where μ_0 is an integration constant.

Case B $k = \pm 1$

Again there are three cases to consider of which we have been able to solve only the case $R_1 = A$. In this case equation (7) gives:

$$R_1 = \frac{SS,_{\tau\tau} + (S,_\tau)^2 + 2kS^2}{S^2} = A \tag{47}$$

where again A is an arbitrary constant.

As before the only interesting case is when $k = A$ which leads to the equation:

$$\frac{SS,_{\tau\tau} + (S,_\tau)^2}{S^2} = 0. \tag{48}$$

The solution of this equation is:

$$S(\tau) = S_0\tau^{1/2} \tag{49}$$

where S_0 is a constant of integration.

The dynamical quantities associated with the scale factor (49) are:

$$\mu = \frac{3\,(4k\tau^2 + 1)}{4S_0^2 t^3} \tag{50}$$

$$p = \frac{3 - 4k\tau^2}{4S_0^2 t^3}. \tag{51}$$

If desired, one can compute the equation of state from (50) and (51).

5 Discussion

In general an equation of state requires the following:

1. A metric, which leads to a given Einstein tensor.
2. A class of observers, which define the physical variables for the given Einstein tensor
3. A number of a priori assumptions among the physical variables defined in step 2.

We have proposed that the last step shall be replaced by a geometric equation at the level of the metric (or any other appropriate geometric object), so that the equation(s) of state for a given metric and a given class of observers will be compatible with the geometric structure of spacetime and will follow in a systematic way from a common assumption. This makes possible the comparison of the equations of state, and consequently the Physics, of different classes of observers in a given spacetime. We have applied this proposal to the highly symmetric FLRW spacetime for the comoving observers the extra geometric assumptions being that the metric admits a RC or a MC. We have obtained linear and non-linear equations of state, which have a sound physical meaning, in the sense that the resulting models satisfy the basic requirements of a viable cosmological model.

References

[1] Wainwright J and Ellis G F R (Editors), *Dynamical Systems in Cosmology* (Cambridge University Press, Cambridge 1997)

[2] King A R and Ellis G F R, *Tilted Homogeneous Cosmological Models*, 1973 Commun. Math. Phys. **31** 209-242

[3] Coley A A and Tupper O J, *Zero-Curvature Friedmann-Robertson-Walker models as exact viscous magnetohydrodynamic cosmologies*, 1983 Astrophys. J. **271**, 1 - 8

[4] McIntosh C B G, *Homothetic motions in general relativity*, 1975 Gen. Relativ. Grav. **7** 199-213

[5] Wainwright J, *Self-similar solutions of Einstein's equations*, in *Galaxies, Axisymmetric Systems and Relativity*, ed. MacCallum M A H (Cambridge University Press, Cambridge 1985)

[6] Katzin G H, Levine J and Davis W R, *Curvature Collineations: A fundamental Symmetry Property of the spacetimes of General Relativity Defined by the Vanishing Lie Derivative of the Riemann Curvature Tensor*, 1969 J. Math. Phys. **10** 617-629

[7] Katzin G H and Levine J, *Applications of Lie derivatives to symmetries, geodesic mappings, and first integrals in Riemannian spaces*, 1972 Colloquium Mathematicum **XXVI** 21-38

[8] Maartens R and Maharaj S D, *Conformal Killing vectors in Robertson-Walker spacetimes*, 1986 Class. Quantum Grav. 3 1005-1011

[9] Maharaj S D and Maartens R, *General Solution of Liouville's Equation in Robertson-Walker spacetimes*, 1987 Gen. Relativ. Grav. **19** 1217-1222

[10] Green L H, Norris L K, Oliver D R and Davis W R, *The Robertson-Walker metrics and the Symmetries Belong to the Family of Contracted Ricci Collineations*, 1977 Gen. Relativ. Gravit. **8,** 731-737

[11] Apostolopoulos P S and Tsamparlis M, *Comment on Ricci Collineations for spherically symmetric spacetimes*, 2002 Gen. Relativ. Gravit. **34** 49-52

[12] Apostolopoulos P S and Tsamparlis M, *Ricci and Matter inheritance collineations in Robertson-Walker spacetimes*, 2001 **gr-qc/0110042**

[13] Camci U and Barnes A, *Ricci collineations in Friedmann-Robertson-Walker spacetimes*, 2002 Class. Quantum Grav. **19** 393-404

[14] Kramer D, Stephani H, MacCallum M A H and Herlt E, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge 1980)

[15] Hawking S W and Ellis G F R, *The large scale structure of spacetime* (Cambridge University Press, Cambridge 1973)